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THE NATURAL FREQUENCIES AND MODE SHAPES OF A UNIFORM CANTILEVER BEAM WITH MULTIPLE TWO-DOF SPRING–MASS SYSTEMS

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Because of the complexity of the mathematical expressions, the literature concerning the free vibration analysis of a uniform beam carrying a "single" two degrees-of-freedom (d.o.f.) spring–mass system is rare and the publications relating to that carrying "multiple" two-d.o.f. spring–mass systems have not yet appeared. Hence the purpose of this paper is to present some information in this area. First of all, the closed form solution for the natural frequencies and the corresponding normal mode shapes of the uniform beam alone (or the "bare" beam) with the prescribed boundary conditions are determined analytically. Next, a method is presented to replace each two-d.o.f. spring–mass system by two massless equivalent springs with spring constants $k^{(v)}_{eq,i}$ and $k^{(v)}_{eq,k}$, and then the foregoing natural frequencies and normal mode shapes for the"bare" beam are in turn used to derive the equation of motion of the "loading" beam (i.e., the bare beam carrying any number of two-d.o.f. spring–mass systems) by using the expansion theorem. Finally, the natural frequencies and the associated mode shapes of the"loading" beam are obtained from the last equation by using the numerical method. To confirm the reliability of the present method, all the numerical results obtained in this paper are compared with the corresponding ones obtained from the conventional finite element method (FEM) and good agreement is achieved. Because the order of the property matrices for the equation of motion of the"loading" beam derived from the present method is much lower than that derived from the FEM, the computer time required by the former is much less than that required by the latter. Besides, the equation of motion derived from the present method may always run on the cheaper personal computers, but that from the FEM may run only on the more expensive larger computers if the degree of freedom of the loading beam exceeds a certain limit.

1. INTRODUCTION

The effects of vibration absorbers on the vehicle suspensions or the rotating machineries and the dynamic behavior of components of machine tool structures due to excitations are the important information that the machine designers hope to obtain. Since beams carrying one or two-degrees-of-freedom (d.o.f.) spring–mass systems are good examples to provide this information, many researchers have

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devoted themselves to the studies of this area. For example, Snowdon [1], Jacquot [2], Dowell [3], Nicholson and Bergman [4]. Özguven and Candir [5], and Manikanahally and Crocker [6] have studied the effects of single- and multiple-d.o.f. spring–mass absorbers. Laura *et al.* [7], Ercoli and Laura [8], Larrondo *et al.* [9], and Gúŕgóze [10] have investigated the behavior of beams carrying a single-d.o.f. spring–mass system. Besides, Frýba [11], Hino *et al.* [12, 13], Yoshimura *et al.* [14, 15] and Lin *et al.* [16, 17] have investigated the vibration problem of single and multiple spring–mass systems moving along a beam by considering the interactions between the suspension systems and the beam. In 1993, Jen and Magrab [18] presented an"exact" solution for the natural frequencies and mode shapes for beams carrying a"single" two-d.o.f. spring–mass system.

In references [16, 17] the governing equations for the"entire" system (i.e., the beam together with the two-d.o.f. spring–mass system) were derived based on the finite element formulation and in references [18, 19] the same work was done by using the Laplace transform with respect to the spatial variable. Instead of the "entire" system, the two-d.o.f. spring–mass system alone is considered as a finite element and then the element property matrices were derived based on the force (and moment) equilibrium between the spring–mass system and the beam in this paper. Besides, it was found in this paper that any two-d.o.f. spring–mass system may be replaced by four massless "effective" springs with spring constants $k^{(v)}_{eff,\,ij}$ $(i, j = 1{,}2)$ or two massless "equivalent" springs with spring constants $k^{(v)}_{eq,\,i}$ and $k^{(v)}_{eq,\,k}$, so that a beam carrying any number of two-d.o.f. spring–mass systems may be considered as a beam supported by a number of effective (or equivalent) springs and the other alternative approaches may be used to solve the same problem in addition to the conventional finite element method (FEM).

In reference [20] alternative formulations of the frequency equation of a Bernoulli–Euler beam carrying several spring–mass systems were presented. But all the spring–mass systems considered were single-degree-of-freedom (d.o.f) systems rather than two-d.o.f. ones studied in this paper. From reference [21] one sees that the order of the property matrices for the equation of motion of a uniform beam carrying multiple concentrated elements derived from the analytical-and-numerical-combined method (ANCM) is much less than that derived from the FEM; therefore, the computing time required by the ANCM is much less than that required by the FEM. For this reason, in this paper ANCM is used to determine the natural frequencies and mode shapes of a uniform cantilever beam carrying multiple two-d.o.f. spring–mass systems. It is noted that all the spring–mass systems studied in reference [21] are those with "single" d.o.f.; hence, this paper is the first to do the free vibration analysis of a uniform beam carrying multiple "two" d.o.f. spring–mass systems by using the ANCM.

To agree with the existing literature [21], a uniform beam with prescribed boundary conditions is called the "constrained" beam if it carries any spring–mass systems, and is called the "unconstrained" beam if it carries nothing. For convenience, they were also called the "loading" beam and "bare" beam in this paper respectively.

2. EQUATION OF MOTION FOR EACH TWO-DOF SPRING–MASS SYSTEM

For an arbitrary two-d.o.f. spring–mass system mounted on a uniform beam at $x = x_i^{(v)}$ and $x = x_k^{(v)}$ as shown in Figure 1(a), the coupled equations for the spring–mass system are defined by

$$\sum F_y = F_i + F_k + F_v - m_e^{(v)}\ddot{u}_v = 0, \tag{1}$$

$$\sum M_z = -F_i a_1^{(v)} + F_k a_2^{(v)} + M_v - J_e^{(v)}\ddot{\theta}_v = 0, \tag{2}$$

where F_v and M_v are the external force and moment on the mass of the spring–mass system, while F_i and F_k are the interactive forces between the two-d.o.f. spring–mass system and the uniform beam at the two attaching points ⓘ and ⓚ. The latter are given by

$$F_i = k_y^{(v)}(u_i - u_v + a_1^{(v)}\theta_v), \qquad F_k = k_y^{(v)}(u_k - u_v - a_2^{(v)}\theta_v). \tag{3, 4}$$

The substitution of equations (3) and (4) into equations (1) and (2) gives

$$m_e^{(v)}\ddot{u}_v + k_y^{(v)}[-u_i - u_k + 2u_v - (a_1^{(v)} - a_2^{(v)})\theta_v] = F_v, \tag{5}$$

$$J_e^{(v)}\ddot{\theta}_v + k_y^{(v)}[a_1^{(v)}u_i - a_2^{(v)}u_k - (a_1^{(v)} - a_2^{(v)})u_v + (a_1^{(v)^2} + a_2^{(v)^2})\theta_v] = M_v. \tag{6}$$

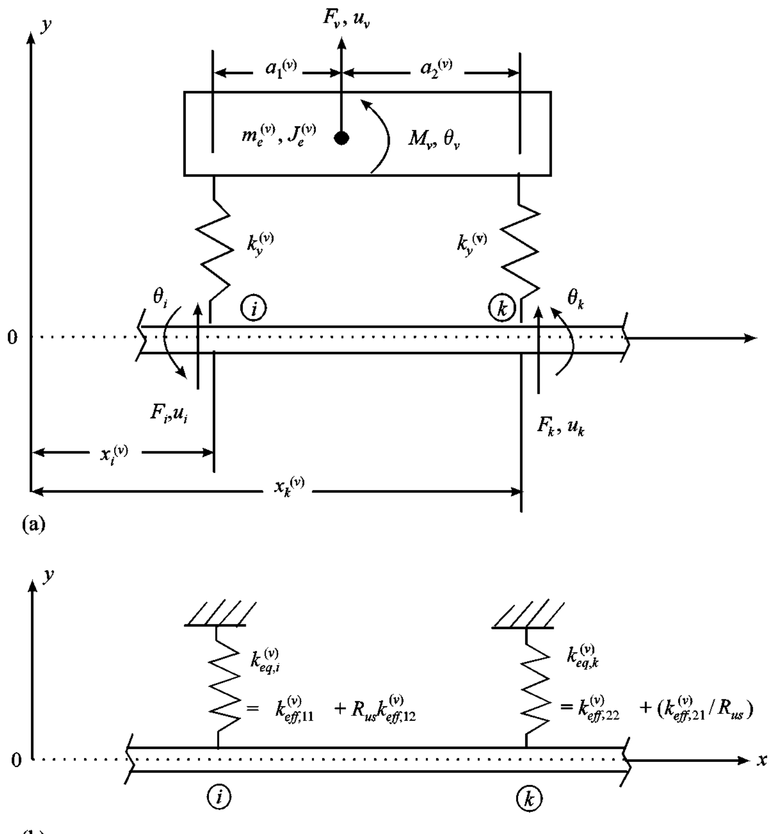


Figure 1(a). A uniform beam carrying an arbitrary two-d.o.f. spring–mass system and (b) the two-d.o.f. spring–system is replaced by two equivalent springs $k_{eq,i}^{(v)}$ and $k_{eq,k}^{(v)}$.

Writing equations (3) – (6) in matrix form, one obtains

$$[m]\{\ddot{u}\} + [k]\{u\} = \{F\}, \tag{7}$$

where

$$[m] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & m_e^{(v)} & 0 \\ 0 & 0 & 0 & J_e^{(v)} \end{bmatrix}, \tag{8}$$

$$[k] = k_y \begin{bmatrix} 1 & 0 & -1 & a_1^{(v)} \\ 0 & 1 & -1 & -a_2^{(v)} \\ -1 & -1 & 2 & -(a_1^{(v)} - a_2^{(v)}) \\ a_1^{(v)} & -a_2^{(v)} & -(a_1^{(v)} - a_2^{(v)}) & (a_1^{(v)^2} + a_2^{(v)^2}) \end{bmatrix}, \tag{9}$$

$$\{u\} = \{u_i \;\; u_k \;\; u_v \;\; \theta_v\}, \qquad \{\ddot{u}\} = \{\ddot{u}_i \;\; \ddot{u}_k \;\; \ddot{u}_v \;\; \ddot{\theta}_v\}, \qquad \{F\} = \{F_i \;\; F_k \;\; F_v \;\; M_v\}.$$

(10a, 10b, 11)

In the last expressions, $[m]$ is the mass matrix, $[k]$ is the stiffness matrix and $\{F\}$ is the external loading vector, while $\{u\}$ and $\{\ddot{u}\}$ are the displacement vector and acceleration vector respectively.

3. EQUIVALENT SPRING CONSTANTS

For the free vibration of spring–mass system, the external loads on it are zero: i.e.,

$$F_v = M_v = 0. \tag{12}$$

In such a case, from equations (5) and (6) one obtains

$$-m_e^{(v)}\ddot{u}_v = k_y^{(v)}(-u_i - u_k) + k_y^{(v)}[2u_v - (a_1^{(v)} - a_2^{(v)})\theta_v], \tag{13}$$

$$-J_e^{(v)}\ddot{\theta}_v = k_y^{(v)}(a_1^{(v)}u_i - a_2^{(v)}u_k) + k_y^{(v)}[-(a_1^{(v)} - a_2^{(v)})u_v + (a_1^{(v)^2} + a_2^{(v)^2})\theta_v]. \tag{14}$$

For free vibration of the constrained beam (i.e., the uniform beam together with the spring–mass system), one has

$$u_v(t) = \bar{u}_v e^{i\bar{\omega}t}, \quad \theta_v(t) = \bar{\theta}_v e^{i\bar{\omega}t}, \tag{15}$$

$$u_i(t) = \bar{u}_i e^{i\bar{\omega}t}, \quad u_k(t) = \bar{u}_k e^{i\bar{\omega}t}, \quad F_i(t) = \bar{F}_i e^{i\bar{\omega}t}, \quad F_k(t) = \bar{F}_k e^{i\bar{\omega}t}, \tag{16}$$

where $\bar{\omega}$ is the natural frequency of the constrained beam, while $\bar{u}_v$, $\bar{\theta}_v$, $\bar{u}_i$, $\bar{u}_k$, $\bar{F}_i$ and $\bar{F}_k$ represent the vibration amplitudes of $u_v(t)$, $\theta_v(t)$, $u_i(t)$, $u_k(t)$, $F_i(t)$ and $F_k(t)$

respectively. Substituting equations (15) and (16) into equations (13) and (14), one obtains

$$\begin{bmatrix} m_e^{(v)}\bar{\omega}^2 & 0 \\ 0 & J_e^{(v)}\bar{\omega}^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_v \\ \bar{\theta}_v \end{Bmatrix} = k_y^{(v)} \begin{bmatrix} -1 & -1 \\ a_1^{(v)} & -a_2^{(v)} \end{bmatrix} \begin{Bmatrix} \bar{u}_i \\ \bar{u}_k \end{Bmatrix} + k_y^{(v)} \begin{bmatrix} 2 & -(a_1^{(v)} - a_2^{(v)}) \\ -(a_1^{(v)} - a_2^{(v)}) & (a_1^{(v)^2} + a_2^{(v)^2}) \end{bmatrix} \begin{Bmatrix} \bar{u}_v \\ \bar{\theta}_v \end{Bmatrix}, \quad (17)$$

or

$$\begin{Bmatrix} \bar{u}_v \\ \bar{\theta}_v \end{Bmatrix} = \begin{bmatrix} m_e^{(v)}\bar{\omega}^2 - 2k_y^{(v)} & (a_1^{(v)} - a_2^{(v)})\,k_y^{(v)} \\ (a_1^{(v)} - a_2^{(v)})\,k_y^{(v)} & J_e^{(v)}\bar{\omega}^2 - (a_1^{(v)^2} + a_2^{(v)^2})\;k_y^{(v)} \end{bmatrix}^{-1} \cdot k_y^{(v)} \begin{bmatrix} -1 & -1 \\ a_1^{(v)} & -a_2^{(v)} \end{bmatrix} \begin{Bmatrix} \bar{u}_i \\ \bar{u}_k \end{Bmatrix}. \quad (18)$$

Introducing equations (15) and (16) into equations (3) and (4) and then writing the results in matrix form, one obtains

$$\begin{Bmatrix} \bar{F}_i \\ \bar{F}_k \end{Bmatrix} = k_y^{(v)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{u}_i \\ \bar{u}_k \end{Bmatrix} + k_y^{(v)} \begin{bmatrix} -1 & a_1^{(v)} \\ -1 & -a_2^{(v)} \end{bmatrix} \begin{Bmatrix} \bar{u}_v \\ \bar{\theta}_v \end{Bmatrix}. \quad (19)$$

The substitution of equation (18) into equation (19) leads to

$$\begin{Bmatrix} \bar{F}_i \\ \bar{F}_k \end{Bmatrix} = \left(k_y^{(v)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k_y^{(v)^2} \begin{bmatrix} -1 & a_1^{(v)} \\ -1 & -a_2^{(v)} \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ a_1^{(v)} & -a_2^{(v)} \end{bmatrix} \right) \begin{Bmatrix} \bar{u}_i \\ \bar{u}_k \end{Bmatrix}, \quad (20)$$

where

$$W_{11} = [J_e^{(v)}\bar{\omega}^2 - (a_1^{(v)^2} + a_2^{(v)^2})\,k_y^{(v)}]/\Delta, \quad W_{22} = (m_e^{(v)}\bar{\omega}^2 - 2k_y^{(v)})/\Delta,$$
$$W_{12} = -(a_1^{(v)} - a_2^{(v)})\,k_y^{(v)}/\Delta = W_{21}, \quad (21)$$

and

$$\Delta = \begin{vmatrix} m_e^{(v)}\bar{\omega}^2 - 2k_y^{(v)} & (a_1^{(v)} - a_2^{(v)})\,k_y^{(v)} \\ (a_1^{(v)} - a_2^{(v)})\,k_y^{(v)} & J_e^{(v)}\bar{\omega}^2 - (a_1^{(v)^2} + a_2^{(v)^2})\,k_y^{(v)} \end{vmatrix}$$
$$= m_e^{(v)} J_e^{(v)}\bar{\omega}^4 - [2J_e^{(v)} + m_e^{(v)}(a_1^{(v)^2} + a_2^{(v)^2})]\,k_y^{(v)}\bar{\omega}^2 + k_y^{(v)^2}(a_1^{(v)} + a_2^{(v)})^2. \quad (22)$$

From equation (20) one sees that the relationship between $\{\bar{F}_i\ \bar{F}_k\}$ and $\{\bar{u}_i\ \bar{u}_k\}$ takes the form

$$\begin{Bmatrix} \bar{F}_i \\ \bar{F}_k \end{Bmatrix} = \begin{bmatrix} k_{eff,11}^{(v)} & k_{eff,12}^{(v)} \\ k_{eff,21}^{(v)} & k_{eff,22}^{(v)} \end{bmatrix} \begin{Bmatrix} \bar{u}_i \\ \bar{u}_k \end{Bmatrix}, \quad (23)$$

where

$$k_{eff,11}^{(v)} = k_y^{(v)} + k_y^{(v)^2}\,(W_{11} - 2a_1^{(v)} W_{12} + a_1^{(v)^2} W_{22})$$

$$k_{eff,12}^{(v)} = k_y^{(v)^2}\,[W_{11} - (a_1^{(v)} - a_2^{(v)})\, W_{12} - a_1^{(v)} a_2^{(v)}\, W_{22}] = k_{eff,\,21}^{(v)}, \qquad (24)$$

$$k_{eff,\,22}^{(v)} = k_y^{(v)} + k_y^{(v)^2}\,(W_{11} + 2a_2^{(v)} W_{12} + a_1^{(v)^2} W_{22}).$$

Equation (23) means that the two-d.o.f. spring–mass system shown in Figure 1(a) may be replaced by four effective springs with spring constants $k_{eff,\,ij}^{(v)}$ $(i, j = 1{,}2)$ (c.f. Figure 1(b)]

In order to apply the analytical-and-numerical-combined method (ANCM) to solve the title problem, the last four "effective" springs must be further replaced by two "equivalent" springs with spring constants $k_{eq,i}^{(v)}$ and $k_{eq,\,k}^{(v)}$. The latter are found to be

$$k_{eq,i}^{(v)} = k_{eff,11}^{(v)} + R_{us} k_{eff,12}^{(v)}, \quad k_{eq,k}^{(v)} = k_{eff,22}^{(v)} + (k_{eff,21}^{(v)}/R_{us}), \qquad (25,\ 26)$$

where

$$R_{us} = u_k/u_i = \bar{Y}_s\,(x_k^{(v)})/\bar{Y}_s\,(x_i^{(v)}). \qquad (27)$$

The last expressions were derived from equation (23). The symbols $\bar{Y}_s\,(x_i^{(v)})$ and $\bar{Y}_s\,(x_k^{(v)})$ in equation (27) represent the values of the modal displacements of the sth mode shape at the attaching points of the vth two-d.o.f. spring–mass system with the beam, $x_i^{(v)}$ and $x_k^{(v)}$ respectively. In reference [22], it has been found that the exact solution presented in reference [18] is correct only if the effects of the coupling effective spring constants $k_{eff,12}^{(v)}$ and $k_{eff,21}^{(v)}$ defined by equation (24) are negligible.

4. EQUATION OF MOTION FOR THE CONSTRAINED BEAM

Once all the two-d.o.f. spring–mass systems are replaced by the equivalent springs with spring constants $k_{eq,\,j}^{(v)}$ $(j = 1, 2$ and $v = 1 \sim p)$, then from equation (12) of reference [21] one may infer that the equation of motion for uniform beam carrying p two-d.o.f. spring–mass systems is given by

$$\int_0^L \sum_{r=1}^{n'} \bar{Y}_s(x)\, EI\, \bar{Y}_T''''\,(x)\, q_r(t)\, \mathrm{d}x + \int_0^L \sum_{r=1}^{n'} \bar{Y}_s(x) \bar{m}\, \bar{Y}_r(x)\, \ddot{q}_r(t)\, \mathrm{d}x$$

$$= -\int_0^L \sum_{v=1}^{p} k_{eq,\,i}^{(v)}\, \bar{Y}_s\,(x_i^{(v)}) \sum_{r=1}^{n'} \bar{Y}_r\,(x_i^{(v)})\, q_r(t)\, \mathrm{d}x$$

$$-\int_0^L \sum_{v=1}^{p} k_{eq,\,k}^{(v)}\, \bar{Y}_s\,(x_k^{(v)}) \sum_{r=1}^{n'} \bar{Y}_r\,(x_k^{(v)})\, q_r(t)\, \mathrm{d}x, \quad s = 1 \sim n', \qquad (28)$$

where

$$\bar{Y}_s(x_i^{(v)}) = \int_0^L \bar{Y}_s\,(x)\, \delta(x - x_i^{(v)})\, \mathrm{d}x, \quad \bar{Y}_s(x_k^{(v)}) = \int_0^L \bar{Y}_s\,(x)\, \delta(x - x_k^{(v)})\, \mathrm{d}x. \quad (29\mathrm{a,b})$$

In the above equations, E is the Young's modulus of the beam material, I is the moment of inertia of the cross-sectional area of the beam, $\bar{m}$ is the mass per unit length of the beam, $\bar{Y}_r(x)$ and $\bar{Y}_s(x)$ are the normal mode shapes of the unconstrained beam, $q_r(t)$ is the generalized co-ordinate, $x_i^{(v)}$ and $x_k^{(v)}$ are the co-ordinates of the attaching points ⓘ and ⓚ [see Figure 1(a)] for the v-th spring–mass system, $\delta(\cdot)$ is the Dirac delta function, and n' is the total number of normal mode shapes of the unconstrained beam considered.

By using the orthogonality of the normal mode shapes $\bar{Y}_r(x)$ and $\bar{Y}_s(x)$, equation (28) reduces to

$$\ddot{q}_s(t) + \omega_s^2\, q_s(t) = -\sum_{v=1}^{p}\sum_{r=1}^{n'} k_{eq,\,i}^{(v)}\, \bar{Y}_r(x_i^{(v)})\, \bar{Y}_s(x_i^{(v)})\, q_s(t) - \sum_{v=1}^{p}\sum_{r=1}^{n'} k_{eq,\,k}^{(v)}\, \bar{Y}_r(x_k^{(v)})\, \bar{Y}_s(x_k^{(v)})\, q_s(t), \tag{30}$$

where ω_s is the sth natural frequency of the unconstrained beam.

For free vibration of the constrained beam, the generalized co-ordinate $q_s(t)$ takes the form

$$q_s(t) = \bar{q}_s\, \mathrm{e}^{i\omega t}, \tag{31}$$

where $\bar{q}_s$ is the amplitude of $q_s(t)$.

The substitution of equation (31) into equation (30) leads to the following equations of motion for the constrained beam:

$$\omega_s^2\, \bar{q}_s + \sum_{v=1}^{p}\sum_{r=1}^{n'} k_{eq,\,i}^{(v)}\, \bar{Y}_r(x_i^{(v)})\, \bar{Y}_s(x_i^{(v)})\, \bar{q}_r + \sum_{v=1}^{p}\sum_{r=1}^{n'} k_{eq,\,k}^{(v)}\, \bar{Y}_r(x_k^{(v)})\, \bar{Y}_s(x_k^{(v)})\, \bar{q}_r = \bar{\omega}^2\, \bar{q}_s, \quad s = 1 \sim n'. \tag{32}$$

5. CHARACTERISTIC EQUATIONS FOR THE CONSTRAINED BEAM

Equation (32) represents a set of n' simultaneous equations. For the convenience of solving the problem, they are rewritten in the matrix form

$$([A] - \bar{\omega}^2\, [B])\, \{\bar{q}\} = \{0\}, \tag{33}$$

where

$$[A]_{n'\times n'} = \lceil \omega^2 \rfloor_{n'\times n'} + [A']_{n'\times n'},\ [B]_{n'\times n'} = \lceil I \rfloor_{n'\times n'},\{\bar{q}\}_{n'\times 1} = [\bar{q}_1\ \bar{q}_2\ \cdots\ \bar{q}_{n'}\}_{n'\times 1},$$

$$[A']_{n'\times n'} = \sum_{v=1}^{p} k_{eq,i}^{(v)}\, [\bar{Y}(x_i^{(v)})]_{n'\times n'} + \sum_{v=1}^{p} k_{eq,k}^{(v)}\, [\bar{Y}(x_k^{(v)})]_{n'\times n'},$$

$$[\bar{Y}(x)]_{n'\times n'} = [\bar{Y}(x)\}_{n'\times 1}\, \{\bar{Y}(x)\}_{n'\times 1}^T,\ \{\bar{Y}(x)\}_{n'\times 1} = \{\bar{Y}_1(x)\ \bar{Y}_2(x)\ \cdots\ \bar{Y}_{n'}(x)\}_{n'\times 1},$$

$$\lceil \omega^2 \rfloor_{n'\times n'} = \lceil\, \omega_1^2\ \omega_2^2\ \cdots\ \omega_{n'}^2 \rfloor_{n'\times n'},\ \lceil I \rfloor_{n'\times n'} = \lceil\, I\ I\ \cdots\ I \rfloor_{n'\times n'}. \tag{34}$$

In the last few expressions, the symbols [ ] , { } and ⌈ ⌋ represents the square matrix, column vector and diagonal matrix, respectively. Non-trival solution of equation (33) requires that

$$|[A] - \bar{\omega}^2 \, [B]| = 0. \tag{35}$$

Since the equivalent spring constant $k^{(v)}_{eq,\,i}$ and $k^{(v)}_{eq,\,k}$ for each two-d.o.f. spring–mass system are functions of the natural frequency $\bar{\omega}$ of the constrained beam as shown by equations (21)–(27), and so are the square matrices $[A']$ and $[A]$ defined by equation (34), the half-interval method [23] was used to solve the eigenvalues $\bar{\omega}_s$ $(s = 1 \sim n')$ and the corresponding eigenvectors $\{\bar{q}\}^{(s)}$ are obtained by substituting the values of $\bar{\omega}_s$ into equation (33). Finally, the mode shapes of the constrained beam are determined by

$$\tilde{Y}_s(x) = \sum_{r=1}^{n'} \bar{Y}_r(x)\, \bar{q}^{(s)}_r = \{\bar{Y}(x)\}^T \{\bar{q}\}^{(s)}, \quad s = 1, 2, \ldots, n'. \tag{36}$$

6. SOLUTION OF THE PROBLEM WITH THE FEM

From the foregoing formulation one also sees that the problem studied in this paper may be solved by two kinds of FEM [22]: the approach considering each two-d.o.f. spring–mass system as a finite element with element property matrices defined by equations (8) and (9) is called Method 1 (or FEM1), and the one replacing each spring–mass system by four effective springs with spring constants $k^{(v)}_{eff,\,ij}$ $(i, j = 1, 2)$ defined by equations (23) and (24) is called Method 2 (or FEM2). The key points of the two methods are stated below. For the details one may refer to reference [22].

6.1. PROPERTY MATRICES FOR FEM1

If the stiffness matrix of the vth two-d.o.f. spring–mass system as shown in Figure 2 is denoted by (see equation (9))

$$\begin{array}{c} \qquad\quad u_{2i-1} \quad\; u_{2k-1} \quad\; u_{2(n+v+1)-1} \quad u_{2(n+v+1)} \\ [k] = \begin{bmatrix} k^{(v)}_{11} & k^{(v)}_{12} & k^{(v)}_{13} & k^{(v)}_{14} \\ k^{(v)}_{21} & k^{(v)}_{22} & k^{(v)}_{23} & k^{(v)}_{24} \\ k^{(v)}_{31} & k^{(v)}_{32} & k^{(v)}_{33} & k^{(v)}_{34} \\ k^{(v)}_{41} & k^{(v)}_{42} & k^{(v)}_{43} & k^{(v)}_{44} \end{bmatrix} \begin{array}{c} u_{2i-1} \\ u_{2k-1} \\ u_{2(n+v+1)-1} \\ u_{2(n+v+1)} \end{array}, \end{array} \tag{37}$$

and the node numbers on the uniform beam at which the two-d.o.f. spring–mass system attached are ⓘ and ⓚ, then the contribution of the element stiffness matrix $[k]$ on the overall stiffness matric $[K]$ is shown in equation (38); where the values of $k_{2i-1,\ 2i-1}$, $k_{2i-1,\ 2k-1}$, $k_{2k-1,\ 2i-1}$ and $k_{2k-1,\ 2k-1}$ are the stiffness coefficients

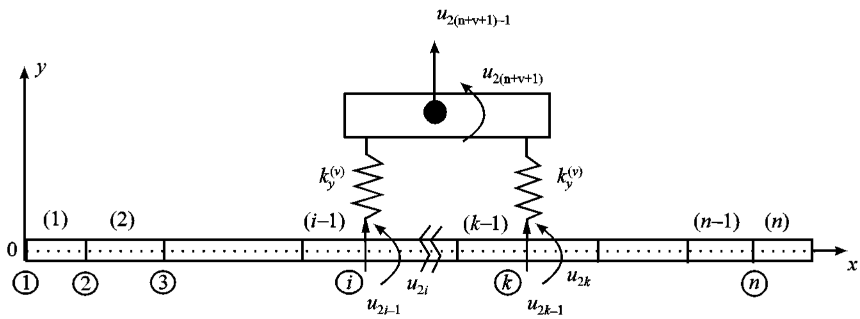


Figure 2. The vth two-d.o.f. spring–mass system attaches the beam at nodes ⓘ and ⓚ.

obtained from assembly of all the uniform beam elements; similarly, the contribution of the element mass matrix $[m]$ to the overall mass matrix $[M]$ may be achieved in a similar way, it is evident that the two attaching points ⓘ and ⓚ are arbitrary and not necessarily adjacent ones.

$$
\begin{array}{c}
\qquad u_1\; u_2\; \cdots\; u_{2i-1} \quad u_{2i}\; \cdots\; u_{2k-1}\; u_{2k}\; \cdots\; u_{2(n+v+1)-1}\; u_{2(n+v+1)}\; \cdots \\
[K] = \left[\begin{array}{cccccc}
 & & & & & \\
k_{2i-1,\,2i-1} + k_{11}^{(v)} & & k_{2i-1,\,2k-1} + k_{12}^{(v)} & & k_{13}^{(v)} & k_{14}^{(v)} \\
 & & & & & \\
k_{2k-1,\,2i-1} + k_{21}^{(v)} & & k_{2k-1,\,2k-1} + k_{22}^{(v)} & & k_{23}^{(v)} & k_{24}^{(v)} \\
k_{31}^{(v)} & & k_{32}^{(v)} & & k_{33}^{(v)} & k_{34}^{(v)} \\
k_{41}^{(v)} & & k_{42}^{(v)} & & k_{43}^{(v)} & k_{44}^{(v)} \\
 & & & & &
\end{array}\right]
\begin{array}{l}
u_1 \\ u_2 \\ \vdots \\ u_{2i-1} \\ u_{2i} \\ \vdots \\ u_{2k-1} \\ u_{2k} \\ \vdots \\ u_{2(n+v+1)-1} \\ u_{2(n+v+1)} \\ \vdots
\end{array}
\end{array}
\tag{38}
$$

It is noted that, in Figure 2, the digits (1, 2, …) in the parentheses ( ) represent the numbers for the finite elements and those in the small circles ○ present those for the nodes.

6.2. PROPERTY MATRICES FOR FEM2

If the uniform beam shown in Figure 2 is a cantilever beam with left end clamped, then the overall stiffness matrix of the beam after imposing the boundary conditions takes the form of equation (39) (excluding the quantities in the parentheses).

It is noted that the identification numbers for the degrees of freedom of the two attached points ⓘ and ⓚ as shown in Figure 2 are changed from $2i$-1, $2i$, $2k$-1 and $2k$ (see equation (38)] to $2i$-3, $2i$-2, $2k$-3 and $2k$-2 (see Equation (39)), respectively.

For convenience, if the effective spring-constant matrix of the vth two-d.o.f. spring–mass system as shown in Figure 1 is represented by equation (40), then the contribution of the element stiffness matrix $[k_{eff}]$ to the overall stiffness matrix $[\bar{K})$ is shown in equation (39); since each two-d.o.f. spring–mass system is replaced by the four massless effective springs with spring constant $k^{(v)}_{eff,\,ij}$ $(i, j = 1, 2)$, the overall mass matrix $[\bar{M}]$ is not affected by the spring masses:

$$
\begin{array}{c|ccccccccc|}
 & \bar{u}_1 & \bar{u}_2 & \cdots & \bar{u}_{2i-3} & \bar{u}_{2i-2} & \cdots & \bar{u}_{2k-3} & \bar{u}_{2k-2} & \cdots \; \bar{u}_{2n}
\end{array}
$$

$$
[\bar{K}] = \begin{bmatrix}
 & & & \\
 & \bar{k}_{2i-3,\,2i-3} \; (+k^{(v)}_{eff,\,11}) & & \bar{k}_{2i-3,\,2k-3} \; (+k^{(v)}_{eff,\,12}) & \\
 & & & \\
 & \bar{k}_{2k-3,\,2i-3} \; (+k^{(v)}_{eff,\,21}) & & \bar{k}_{2k-3,\,2k-3} \; (+k^{(v)}_{eff,\,22}) & \\
 & & &
\end{bmatrix}
\begin{matrix} \bar{u}_1 \\ \bar{u}_2 \\ \cdots \\ \bar{u}_{2i-3} \\ \bar{u}_{2i-2} \\ \vdots \\ \bar{u}_{2k-3} \\ \bar{u}_{2k-2} \\ \vdots \\ \bar{u}_{2n} \end{matrix}, \tag{39}
$$

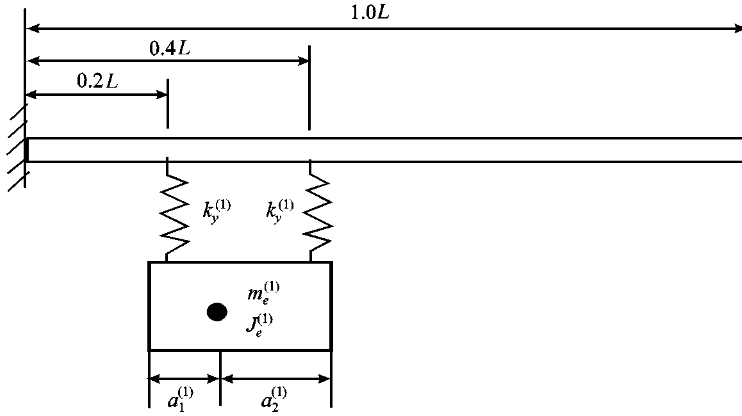
$$
[k_{eff}] = \begin{bmatrix} k^{(v)}_{eff,\,11} & k^{(v)}_{eff,\,12} \\ k^{(v)}_{eff,\,21} & k^{(v)}_{eff,\,22} \end{bmatrix}. \tag{40}
$$

7. NUMERICAL RESULTS AND DISCUSSIONS

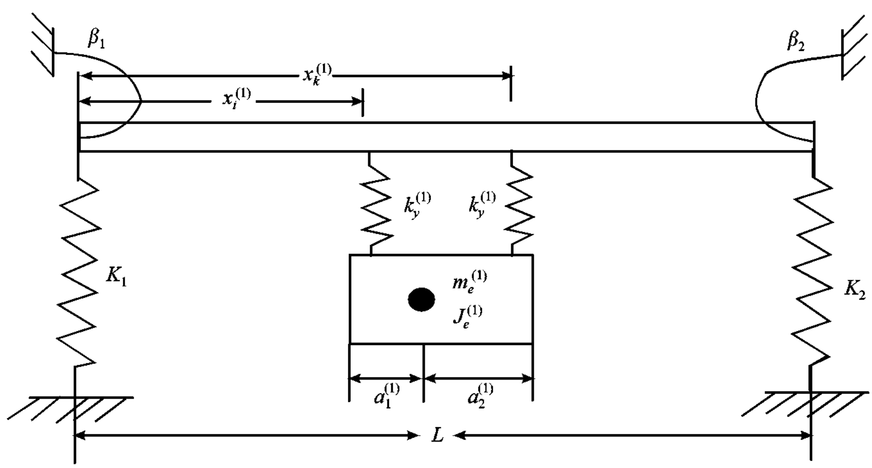
The dimensions and the material constants for the uniform beam studied in this paper (except the one shown in Figure 3(b) of section 7.1) are total length $L = 1{\cdot}0$ m, diameter $d = 0{\cdot}05$ m, Young's modulus $E = 2{\cdot}069 \times 10^{11}$ N/m^2, mass density $\rho = 7{\cdot}8367 \times 10^3$ kg/m^3, mass per unit length $m = \rho A = 15{\cdot}3875$ kg/m, cross-sectional area $A = \pi d^2/4 = 1{\cdot}9635 \times 10^{-3}$ m^2, moment of inertia $I = \pi d^4/64 = 3{\cdot}06796 \times 10^{-7}$ m^4, total mass of the beam $m_b = mL = 15{\cdot}3875$ kg.

7.1. RELIABILITY OF THE RESULTS

In order to confirm the reliability of the theory presented in this paper and the computed programs developed based on the related algorithm, a clamped–clamped uniform beam carrying a two-d.o.f spring–mass system located at $x^{(1)}_i = 0{\cdot}2$ m and



(a)



(b)

Figure 3 (a). A clamped–clamped uniform beam carrying a two-d.o.f. spring–mass system ($m_e^{(1)} = 1{\cdot}53875$ kg, $J_e^{(1)} = 1{\cdot}53875$ kg m^2, $k_y^{(1)} = 6{\cdot}34761 \times 10^6$ N/m, $x_i^{(1)} = 0{\cdot}2$ m, $x_k^{(1)} = 0{\cdot}4$ m, $L = 1{\cdot}0$ m, $a_1^{(1)} = 0{\cdot}06667$ m, $a_2^{(1)} = 0{\cdot}13333$ m). (b) A general restrained uniform beam carrying a two-d.o.f. spring–mass system [19] ($m_e^{(1)} = 500{\cdot}0$ kg, $J_e^{(1)} = 177{\cdot}0833$ kg m^2, $k_y^{(1)} = 1{\cdot}0 \times 10^{10}$ N/m, $x_i^{(1)} = 1{\cdot}4$ m, $x_k^{(1)} = 2{\cdot}6$ m, $L = 4.0$ m, $a_1^{(1)} = 0{\cdot}6$ m, $a_2^{(1)} = 0{\cdot}6$ m, $K_1 = K_2 = 1{\cdot}0 \times 10^{10}$ N/m, $\beta_1 = \beta_2 = 1{\cdot}0 \times 10^{10}$ N m/rad, $E = 2{\cdot}0 \times 10^{11}$ N/m^2, A $= 0{\cdot}15$ m^2, $I = 0{\cdot}003125$ m^4, $\rho = 7860$ kg/m^3).

$x_k^{(1)} = 0{\cdot}4$ m with $a_1^{(1)} = 0{\cdot}06667$ m and $a_2^{(1)} = 0{\cdot}13333$ m presented in Figure 6 of reference [18] and a general restrained uniform beam carrying a two-d.o.f. spring–mass system presented in Figure 1 of reference [19] are studied here (see Figures 3(a) and (b)). For the uniform beam shown in Figure 3(a), besides the dimensions and the material constants mentioned above, the other given data are: $m_e^{(1)} = 0{\cdot}1 \quad \mathrm{m}_b = 1{\cdot}53875$ kg, $k_y^{(1)} = 100(EI/L^3) = 6{\cdot}34761 \times 10^6$ N/m (for each spring) and $J_e^{(1)} = 0{\cdot}1\ (m_b\ L^2) = 1{\cdot}53875$ kg m^2. The first five natural frequencies of the constrained beam are shown in Table 1(a), where are values of $\bar{\omega}_s$ $(s = 1, 2, \ldots, 5)$ listed in the third row were obtained from FEM1 (i.e., the two-d.o.f.

TABLE 1

(a) *The first five natural frequencies* $\bar{\omega}_s$ ($s = 1$–5) *of a uniform clamped–clamped beam carrying a two-d.o.f. spring–mass system obtained from FEM*1, *FEM*2, *ANCM and reference* [18]

Methods		Natural frequencies $\bar{\omega}_s$ (rad/s) $\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	CPU time (s)
FEM 1		269·9453	1385·0940	2849·3420	4215·9160	7837·3880	6
FEM2	(a)*	273·8565	1388·5937	2879·7694	4221·9181	7837·4548	25
	(b)*	253·1247	1542·8917	2823·2835	4162·7274	7841·0863	25
ANCM	(a)*	273·8904	1388·6244	2880·5511	4222·2172	7837·1068	3
	(b)*	253·1590	1542·9125	2823·5460	4162·9047	7840·7393	3
Reference [18]		254·5	—	—	—	—	—
Corresponding natural freq. of the bare beam		—	1436.9860	—	3961·1374	7765·4533	—

* (a) is for the cases of "considering" the effects of the coupling spring constants $k^{(v)}_{eff,12}$ and $k^{(v)}_{eff,21}$.
* (b) is for the cases of "neglecting" the effects of the coupling spring constants $k^{(v)}_{eff,12}$ and $k^{(v)}_{eff,21}$.

(b) *The first five natural frequencies* $\bar{\omega}_s$ ($s = 1$–5) *of a general restrained uniform beam carrying a two-d.o.f. spring–mass system obtain from FEM*1, *FEM*2, *ANCM and reference* [19]

Methods		Natural frequencies $\bar{\omega}_s$ (rad/s) $\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	CPU time (s)
FEM 1		821·9344	1996·4932	3485·5766	4674·3961	6352·8630	6
FEM2	(a)*	821·9344	1996·4932	3485·5766	4674·3961	6352·5000	25
	(b)*	822·4398	1995·3666	3485·6766	4669·2318	6350·0924	25
ANCM	(a)*	821·9406	1996·6119	3485·5879	4677·3631	6352·5000	3
	(b)*	822·4459	1995·4873	3485·6877	4672·2429	6350·1312	3
Reference [19]		822	1995	3585	4669	6348	—
Associated natural freq. of the bare beam		891·1676	2130·5243	3496·3451	5185·5954	7766·5611	—

* (a) is for the cases of "considering" the effects of the coupling spring constants $k^{(v)}_{eff,12}$ and $k^{(v)}_{eff,21}$.
* (b) is for the cases of "neglecting" the effects of the coupling spring constants $k^{(v)}_{eff,12}$ and $k^{(v)}_{eff,21}$.

spring–mass system is considered as a finite element with property matrices defined by equations (8) and (9)) and those listed in the rows 4–7 were obtained from FEM2 and ANCM, respectively. Row 8 lists the first natural frequency obtained from reference [18], $\bar{\omega}_1 = 254{\cdot}5$ rad/s. It is evident that the last value is very close to the first natural frequencies obtained from FEM2(b) and ANCM(b), $\bar{\omega}_1 = 253{\cdot}1$ rad/s.

As shown at the bottom of Tables 1(a), (b), 2 and 4, the natural frequencies $\bar{\omega}_s$ ($s = 1, 2, \ldots$) listed in the rows of FEM2(a) and ANCM(a) were obtained by "considering" the effects of the coupling effective spring constants, $k^{(v)}_{eff,\,12}$ and $k^{(v)}_{eff,\,21}$ and those listed in the rows of FEM2(b) and ANCM(b) were obtained by "neglecting" the last effects. Hence the foregoing agreement between the values of $\bar{\omega}_1$ confirms the reliability of the finding of reference [22] that the formulation of reference [18] was obtained under the assumption that the effects of $k^{(v)}_{eff,\,12}$ and $k^{(v)}_{eff,\,21}$ were neglected.

From Table 1(a) on finds that the values of $\bar{\omega}_s$ ($s = 1, 2, \ldots$) obtained from FEM2(a) and ANCM(a) are very close to the corresponding ones obtained from the conventional finite element method, FEM1. The first five mode shapes obtained from FEM1, FEM2(a) and ANCM(a), $\tilde{Y}_s(\xi_i)$ (s = 1,2, …), are shown in Figures 4(a) (i)–(v), where the mode shapes obtained form the FEM1 are denoted by the solid lines (——), while those obtained from the FEM2(a) and ANCM(a) are denoted by the dash lines (– – –) and dash lines with stars (- - ★- -) respectively. From the last figures one sees the mode shapes obtained from the three methods (FEM1, FEM2(a) and ANCM(a)) almost coincide with each other. The first mode shapes obtained from the FEM2(a) and the FEM2(b) were plotted in Figure 4(a) (vi) to show the effects of $k^{(v)}_{eff,\,12}$ and $k^{(v)}_{eff,\,21}$. Among the two curves of Figure 4(a) (vi), the one with hollow circles (○) was obtained from FEM2(b), which is very close to the corresponding one shown in Figure 6 of reference [18]. It is noted that all the mode shapes shown in Figure 4(a) were "normalized" with respect to the respective maximum modal displacements so that the amplitude of each mode shape is equal to 1·0. Besides, the mode shapes obtained from FEM2 and ANCM are the ones of the beam itself, since each two-d.o.f. spring–mass system was replaced by the four effective springs with spring constants $k^{(v)}_{eff,\,ij}$ (i, j = 1, 2) or by the two equivalent springs with spring constants $k^{(v)}_{eq,\,i}$ and $k^{(v)}_{eq,\,k}$. However, the mode shapes obtained from the FEM1 are the ones of the beam together with the two-d.o.f spring-mass systems, since each spring–mass system occupies two d.o.f. in the FEM1. For this reason, in addition to the modal displacements of the beam itself, the translational movement and the rotational angle of each two-d.o.f. spring–mass system will appear in the modal displacements obtained from the FEM1. Therefore, the curves in Figure 4(a) were plotted by excluding all the modal displacements of the two-d.o.f. spring–mass systems and "double" normalizations for the mode shapes obtained from the FEM1 were required sometimes. For example, in the third mode of the vibrating system shown in Figure 3(a), the largest modal displacement is the translational movement of the lump of the spring–mass system. Hence the third "normal" mode shape obtained from the computer output based on the FEM1 is the thin solid line with amplitude $\tilde{Y}_3$ (0·6) $\cong$ 0·34766 as shown in Figure 4(a) (iii). A good agreement with the other two curves was obtained when a second normalization was made, i.e., all the modal displacements of the thin solid line were divided again by 0·34766. The final result is shown by the wider solid line of Figure 4(a) (iii).

Another example to be used to check the reliability of the theory and the computer programs is the uniform beam supported by two linear springs and two rotational springs at both ends as shown in Figure 3(b). All the given data and the

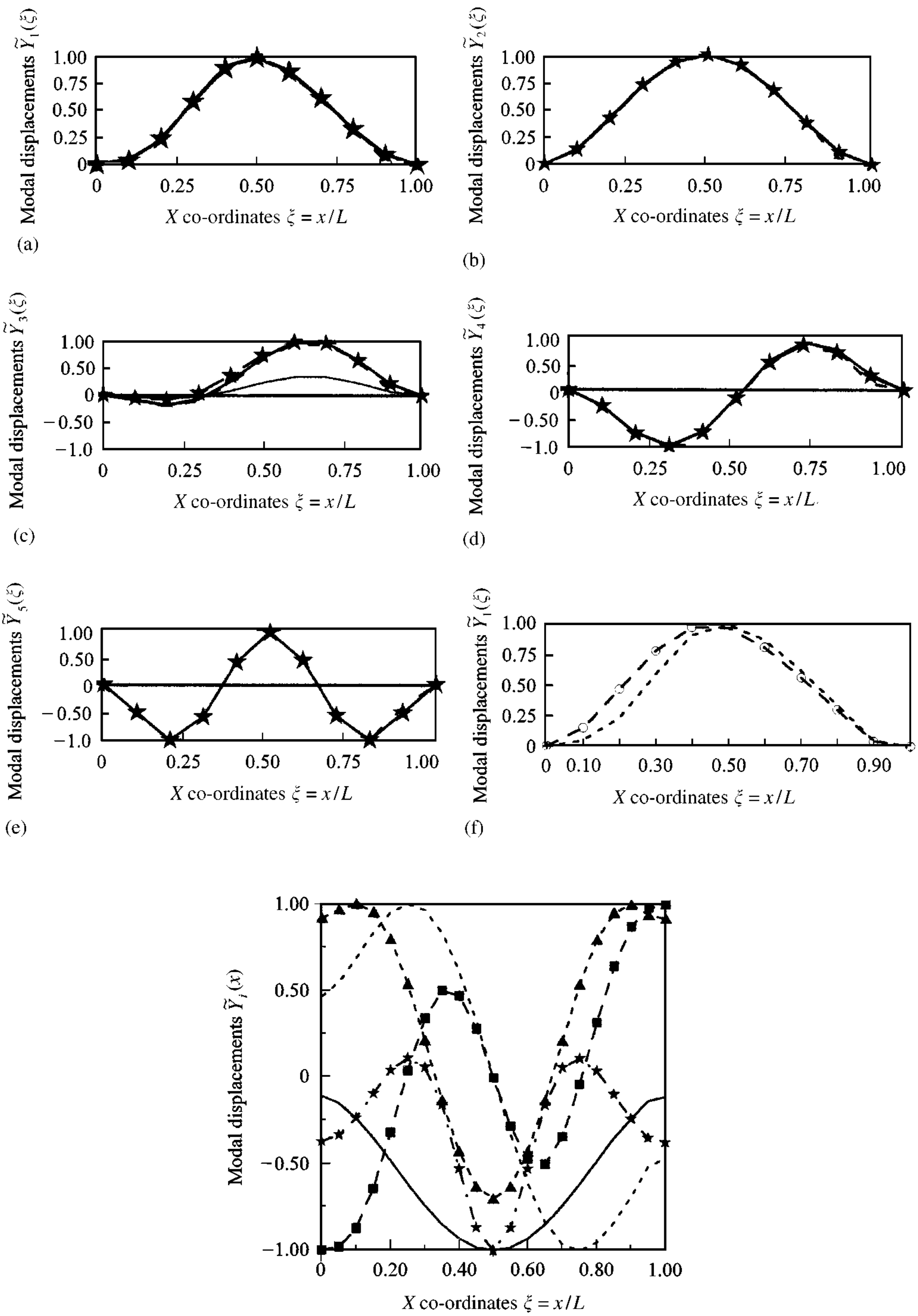


Figure 4 (a). The first five mode shapes for the vibrating system shown in Figure 3 obtained from the FEM1, FEM2 and ANCM: ———, by using FEM1;- - -, by using FEM2 (a): —★—, by using ANCM(a); –○–, by using FEM2(b).) (b). The first five mode shapes for the vibrating system shown in Figure 3(b) obtained from the FEM2(b) or ANCM(b): —, the 1st mode; - - -, the 2nd mode; - -▲- -, the 3rd mode; - -■- -, the 4th mode; - -★- -, the 5th mode.

boundary conditions of the bema are exactly the same as those in referenece [19]. For reference, all the given data are shown in the legend of Figure 3(b). The first five natural frequencies and the asociated mode shapes are shown in Table 1(b) and Figure 4(b). From Table 1(b) and Figure 4(b), one sees that all the first five natural frequencies and the corresponding moe shapes obtained from the FEM2(b) and ANCM(b) are in close agreement with those show on in reference [19].

From all the reasonable facts indicated in this section, it is believed that the results presented in this paper should be reliable.

7.2. NATURAL FREQUENCIES AND MODE SHAPES OF A CANTILEVER BEAM CARRYING A TWO-D.O.F. SPRING–MASS SYSTEM

Figure 5 shows a uniform cantilever beam carrying a two-d.o.f. spring–mass system at the free end. All the dimensions and material constants of the beam and the spring–mass system are exactly equal to those of the last exmple except that the boundary conditions of the beam were changed from clamped–clamped to clamped–free ones and the suspension positions of the two-d.o.f. spring–mass system were also moved from $x_i^{(1)} = 0{\cdot}2L$ and $x_k^{(1)} = 0{\cdot}4L$ to $x_i^{(1)} = 0{\cdot}8L$ and $x_k^{(1)} = 1{\cdot}0L$, where L is the total length of the beam. The first five natural frequencies obtained from the three methods are shown in Table 2. It is evident that the values of $\bar{\omega}_s$ $(s = 1–5)$ obtained from the FEM2(a) and ANCM(a) are very close to the corresponding ones obtained from FEM1. Besides, the values of $\bar{\omega}_s$ $(s = 1–5)$ obtained from the FEM2(b) are also in good agreement with those obtained from ANCM(b). The first five mode shapes obtained from FEM1, FEM2(a) and ANCM(a) are almost overlapped and are shown in Figure 6(b). For convenience of comparison, the first five mode shapes of the unconstrained (bare) uniform cantilever beam are placed in Figure 6(a). For an unconstained (bare) beam, excluding the boundary (supporting) end, the intersections (or nodes) between the

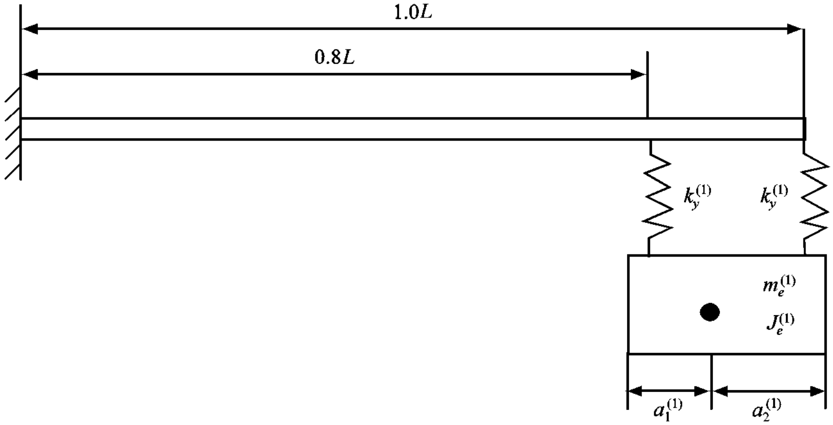


Figure 5. A uniform cantilever beam carrying a two-d.o.f. spring–mass system at the free end.

TABLE 2

The first five natural frequencies $\bar{\omega}_s$ (s = 1–5) of a uniform cantilever beam carrying a two d.o.f. spring–mass system at the free end as shown in Figure 5

Methods		Natural frequencies $\bar{\omega}_s$ (rad/s) $\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	CPU time (s)
FEM 1		141·5405	321·3685	1524·3220	3297·6410	4276·0260	6
FEM2	(a)*	143·4206	324·2268	1526·8963	3326·6748	4279·7603	25
	(b)*	54·5024	949·9614	1606·2917	3168·3310	4316·6266	25
ANCM	(a)*	143·4354	324·3061	1526·9812	3330·0140	4281·0266	3
	(b)*	54·5042	950·2636	1606·3952	3170·7105	4317·6319	3
Corresponding natural freq. of the bare beam		—	—	1415·1955	—	3962·6509	—

* (a) is for the cases of "considering" the effects of the coupling spring constants $k^{(v)}_{eff,12}$ and $k^{(v)}_{eff,21}$.
* (b) is for the cases of "neglecting" the effects of the coupling spring constants $k^{(v)}_{eff,12}$ and $k^{(v)}_{eff,21}$.

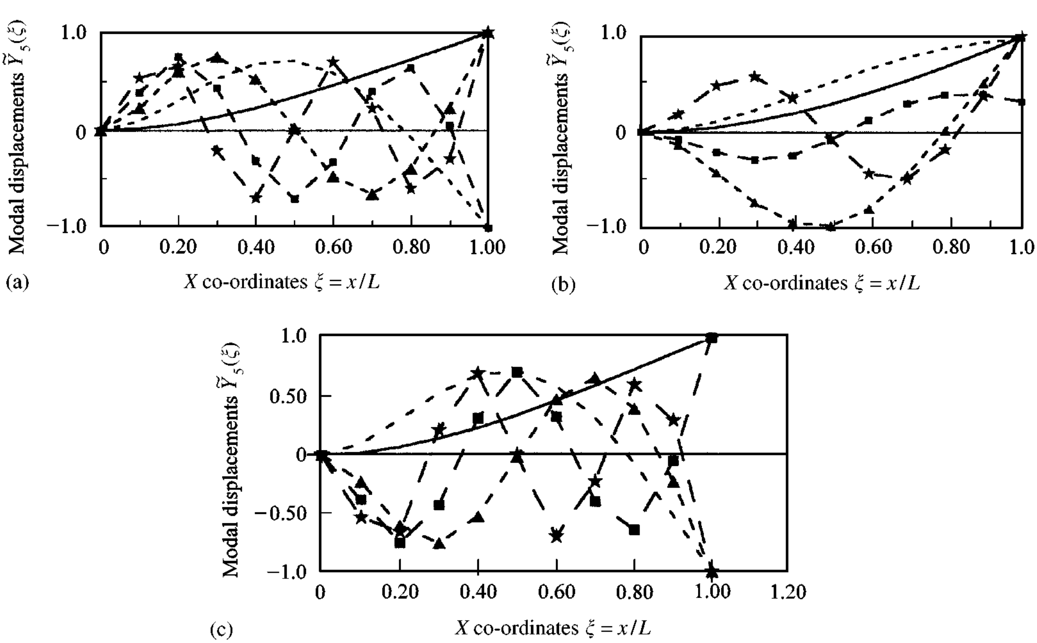


Figure 6. The fist five mode shapes of a uniform cantilever beam in the conditions that: (a) no thing attached; (b) carrying a single two-d.o.f. spring–mass system (se Figure 5); (c) carrying three two-d.o.f. spring–mass systems (see Figure 7) :—, the 1st mode; - - - , the 2nd mode; -▲-, the 3rd mode; - - ■ - -, the 4th mode: - -★ - -, the 5th mode.

ith mode shape and the x-axis are equal to $i-1$. In other words, the nodes for the 1st, 2nd, 3rd, 4th and 5th modes of an "unconstrained" cantilever beam are, respectively, equal to 0, 1, 2, 3 and 4, as one may see from Figure 6(a). But this is not true for the mode shapes of the "constrained" beam shown in Figure 6(b). The final row of Table 2 lists the 1st and 2nd natural frequencies of the unconstrained beam ($\omega_1 = 1415{\cdot}1955$ rad/s, $\omega_2 = 3962{\cdot}6509$ rad/s). It is reasonable that the existence of a single two-d.o.f. spring–mass system located at the free end largely reduces the 1st and 2nd natural frequencies of the cantilever beam ($\bar{\omega}_1 = 141{\cdot}5405$ rad/s, $\bar{\omega}_2 = 321{\cdot}3685$ rad/s).

7.3. NATURAL FREQUENCIES AND MODE SHAPES OF A CANTILEVER BEAM CARRYING THREE TWO-D.O.F. SPRING–MASS SYSTEMS

To show the availability of the present technique for solving the title problem, a uniform cantilever beam carrying three arbitrary two-d.o.f. spring–mass systems as shown in Figure 7 was studied. The dimensions and material constants of the cantilever beam are the same as the foregoing examples, while the locations and the physical properties of the three spring–mass systems are shown in Table 3. The first five natural frequencies $\bar{\omega}_s$ ($s = 1$–5) obtained from the FEM1, FEM2(a), FEM2(b), ANCM(a) and ANCM(b) are displayed in Table 4. As mentioned previously that FEM2(a) and ANCM(a) consider the effects of $k^{(v)}_{eff,\,ij}$ ($i \neq j$), but the last effects were neglected by the FEM2(b) and ANCM(b). The first five mode shapes obtained from the FEM2(a) and ANCM(a) are in good agreement with the corresponding ones obtained from the conventional finite element method (FEM1) and are placed in Figure 6c.

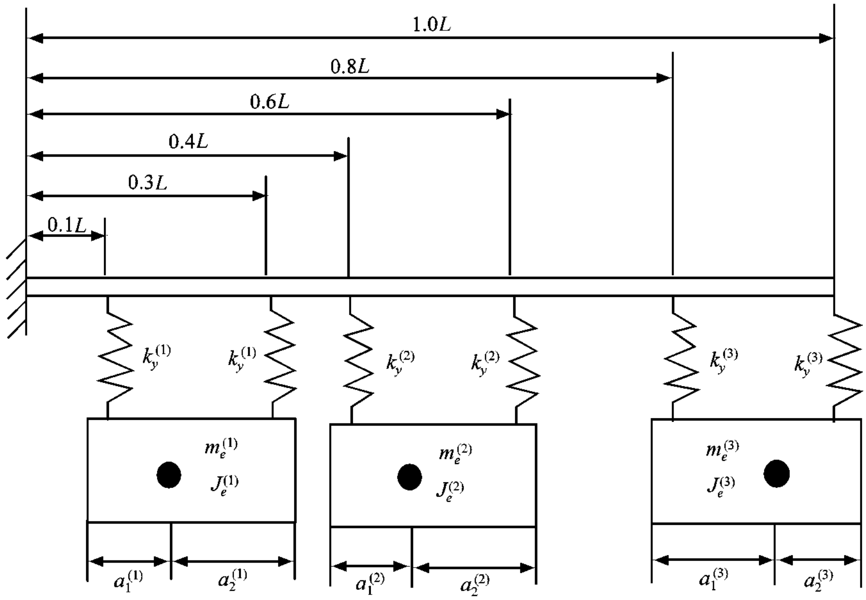


Figure 7. A uniform cantilever beam carrying three two-d.o.f. spring–mass systems.

TABLE 3

The locations and physical properties of the three two-d.o.f. spring–mass systems carried by the uniform cantilever beam shown in Figure 7

Numbering of systems (v)	Locations $\xi_j^{(v)} = x_j^{(v)}/L$ $\xi_i^{(v)}$	$\xi_k^{(v)}$	Physical properties of the spring–mass systems $a_1^{(v)}$ (m)	$a_2^{(v)}$ (m)	$k_y^{(v)}$ (N/m)	$m_e^{(v)}$ (kg)	$J_e^{(v)}$ (kg-m^2)
1	0·1	0·3	0·07	0·13	60·0	1·6	1·6
2	0·4	0·6	0·06	0·14	600·0	1·6	3·2
3	0·8	1·0	0·12	0·08	6000·0	1·6	4·8

TABLE 4

The first five natural frequencies $\bar{\omega}_s$ $(s = 1 \sim 5)$ *of a uniform cantilever beam carrying three two-d.o.f. spring–mass systems as shown in Figure 7*

Methods		Natural frequencies $\bar{\omega}_s$ (rad/s) $\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	CPU time (s)
FEM 1		230·5258	1415·7550	3962·9230	7765·8540	12838·750	8
FEM2	(a)*	231·9355	1415·8251	3962·9617	7765·8580	12838·743	28
	(b)*	231·5426	1415·8249	3962·9617	7765·8580	12838·743	28
ANCM	(a)*	231·9466	1415·7972	3962·8895	7765·4563	12836·626	4
	(b)*	231·5538	1415·7970	3962·8895	7765·4563	12836·626	4
Corresponding natural freq. of the bare beam		225·8360	1415·1955	3962·6509	7565·3119	12836·537	—

* (a) is for the cases of “considering” the effects of the coupling spring constants $k_{eff,12}^{(v)}$ and $k_{eff,21}^{(v)}$.
* (b) is for the cases of “neglecting” the effects of the coupling spring constants $k_{eff,12}^{(v)}$ and $k_{eff,21}^{(v)}$.

From Table 4 one sees that the first five natural frequencies of the “constrained” beam ($\bar{\omega}_1 = 230{\cdot}5$, $\bar{\omega}_2 = 1415{\cdot}8$, $\bar{\omega}_3 = 3962{\cdot}9$, $\bar{\omega}_4 = 7765{\cdot}9$, $\bar{\omega}_5 = 12838{\cdot}8$ rad/s) are very close to those of the “unconstrained” beam listed in the final row of Table 4 ($\omega_1 = 225{\cdot}8$, $\omega_2 = 1415{\cdot}2$, $\omega_3 = 3962{\cdot}7$, $\omega_4 = 7765{\cdot}3$, $\omega_5 = 12836{\cdot}5$ rad/s). From Figures 6(c) and (a) one sees that the first five mode shapes of the “constrained” beam are also quite close to those of the “unconstrained” beam.

By comparing the present results with those of the last sections, one finds that the influence on the dynamic characteristics of the constrained beam due to a single two-d.o.f. spring–mass system is much more than that due to the multiple two-d.o.f. spring–mass systems. This phenomenon seems to be like the effects on a beam due to the action of a concentrated load and the distributed loads. Let $\Delta\bar{\omega}_s = |\bar{\omega}_{sa} - \bar{\omega}_{sb}|$ $(s = 1\text{–}5)$ represent the absolute value of the difference between the sth natural frequency ($\bar{\omega}_{sq}$) obtained from FEM2(a) or ANCM(a) (by “considering” the effects of the coupling effective spring constants $k^{(v)}_{eff,12}$ and $k^{(v)}_{eff,21}$) and the sth one ($\bar{\omega}_{sh}$) obtained from, FEM2(b) or ANCM(b) (by "neglecting" the effects of the coupling effective spring constants $k^{(v)}_{eff,12}$ and $k^{(v)}_{eff,21}$), then from Tables 1, 2, and 4 one sees that the values of $\Delta\bar{\omega}_s$ due to a single two-d.o.f. spring–mass system (as shown in Tables 1 and 2) are also much larger than those due to the multiple spring–mass systems (as shown in Table 4). It is believed that this phenomenon has also something to do with the distribution of the two-d.o.f. spring–mass systems along the length of the carrying beam.

It is also worthy of mention that the CPU time requires by the ANCM is about one half of that required by the FEM1 as one may see from the final columns of Tables 1(a) (b), 2 and 4. The total number of finite elements for each uniform beam was 20, since exstensive studies of the problem show that this finite element model will give the reasonable accuracy for the natural frequencies. Besides, the computing machine used is the IBM 486PC. Since the ANCM is available only if each two-d.o.f. spring–mass system is replaced by the two equivalent springs with spring constants $k^{(v)}_{eq,i}$ and $k^{(v)}_{eq,k}$, the theory presented in this paper should be significant from this point of view.

8. CONCLUSIONS

1. The analytical-and-numerical-combined method (ANCM) is one of the most effective techniques to determine the natural frequencies and the corresponding mode shapes of a uniform beam carrying multiple two degree-of-freedom (d.o.f.) spring–mass systems. The computer time requires by the ANCM is about one-half of that required by the conventional finite element method (FEM1) for the examples illustrated in this paper. However, the ANCM is available only if each two-d.o.f. spring–mass system is replaced by the two equivalent springs with spring constants $k^{(v)}_{eq,i}$ and $k^{(v)}_{eq,k}$ by means of the presented technique.

2. In addition to saving the computer time, the presented approach excluded all the trivial modes associated with the "local vibrations" of all the attached two-d.o.f. spring–mass systems. It is believed that this will be significant if only the dynamic responses of the uniform beam alone are interested. Besides, the presented approach provides an alternative simple unified technique for solving the title problem in addition to the conventional finite element method.

3. In general, the influence on the dynamic characteristics of a uniform beam due to the action of a concentrated load is much more than that due to the distributed loads. This seems also true for a single spring–mass system and the multiple spring–mass systems. Of course, it is believed that the parameters of the two-d.o.f. spring–mass systems (such as $m^{(v)}_e$, $J^{(v)}_e$, $k^{(v)}_y$, $a^{(v)}_1$ and $a^{(v)}_2$) should be also the key factors in addition to the distribution of the two-d.o.f. spring–mass systems along the length of the beam. This will be the topic for our further study.

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